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1990 J. Phys. A: Math. Gen. 23 553

(http://iopscience.iop.org/0305-4470/23/4/023)

J. Phys. A: Math. Gen. 23 (1990) 553-561. Printed in the UK

Supercritical effects and the δ potential

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Received 2 June 1989, in final form 21 September 1989

Abstract. We present a discussion of the occurrence of supercritical effects induced by an attractive δ potential. A detailed analysis of the one-dimensional Dirac equation in the presence of such a potential shows that the ground state never dives into the Dirac sea and therefore the supercritical effects are absent. In the three-dimensional spherically symmetric δ-potential case, in contrast, we find supercritical effects. This situation represents an interesting problem in relativistic quantum mechanics because the δ potential induces discontinuities in the wavefunction, in contrast to the well known Schrödinger case where only the first derivative of the wavefunction becomes discontinuous.

1. Introduction

The aim of this paper is to explore the possibility of inducing supercritical effects by perturbing the Dirac sea with an attractive δ potential. As is well known [1], it is possible to have positron emission induced by the presence of a very strong attractive external potential which dives into the sea of negative-energy electrons.

This situation occurs when the energy associated with an unoccupied bound state of the potential becomes degenerate with a level of the sea. Since the sea consists of a continuum of negative-energy states, an electron belonging to it can be trapped in the potential, leaving a hole with positive energy in the sea, i.e. a positron which escapes up to infinity. This phenomenon, where the appearance of the positron implies a charged vacuum, is called a supercritical effect. The external fields which are responsible for a potential with 'dived' bound states are called supercritical fields. In fact, the external potential must be very strong in order to produce bound states with an energy greater, in absolute value, than $2\, m_e c^2 \approx 1$ MeV. This should be compared, for example, with the 13.6 eV of the ground-state energy of the hydrogen atom.

This idea of supercritical effects seems to play, however, a crucial role in the description of recent exciting heavy-ion scattering experiments [2], where two heavy nuclei, slowly touching each other, produce e^-e^+ pairs with a multiple peak energy spectrum. Although this phenomenon has not been completely explained by theory [3], we do know that when the two nuclei are close enough, a supercritical Coulomb potential, with a dived ground state, will be induced for a sufficiently high value of $Z_1 + Z_2$ ($\geqslant 173$), producing the outgoing positron according to our previous discussion. The mechanism which, in a second stage, is responsible for the outgoing electron is not completely understood. However, some models, like the one by Scharf and Twerenbold [3], claim to reproduce the basic structure of the spectrum. Recently it has been speculated that a new phase of QED, with a soliton-like structure, is produced when supercritical effects take place [4].

0305-4470/90/040553 + 09\$03.50

With these considerations in mind, we have investigated the situation where the Dirac sea is perturbed by the most simple localised potential, a δ potential. Note that the δ potential, constructed as a limit of regular attractive potentials, is 'infinitely' profound and dives into the depths of the Dirac sea. In principle we could expect supercritical effects, and this will be, in fact, the case in the three-dimensional problem. This is not true, however, for the one-dimensional situation.

Although this scenario might be considered as an academic problem, a kind of 'gedanken' experiment, the δ potential constitutes, however, an interesting problem in relativistic quantum mechanics. As far as we know this problem has not been discussed in the literature, as is the case in Schrödinger theory.

Since the Dirac equation is a first-order differential equation in the spacetime variables, a singular δ potential will induce discontinuities in the wavefunction itself, i.e. in the components of the Dirac spinor. Note that in the Schrödinger version of the δ-potential problem [5] there are discontinuities in the first derivative of the wavefunction, because the stationary Schrödinger wave equation is a second-order differential equation.

This paper is organised as follows. In section 2 we analyse the one-dimensional case, showing the absence of supercritical effects for a δ potential with the support at the origin. Section 3 is devoted to the study of the three-dimensional spherically symmetrical δ potential. We show that, in this case, the ground state actually dives into the Dirac sea, for a certain value of the δ coefficient, and therefore we will have supercritical effects. Finally, in section 4, we summarise our conclusions.

2. The one-dimensional case

In this section we will consider a relativistic electron restricted to move in one space dimension in the presence of the attractive δ potential, $V(z) = -a\delta(z)$, where a is a real positive coefficient. This is the most simple case in order to analyse the discontinuity of the wavefunction induced by such a singular potential. The corresponding Dirac equation is given by

$$(\alpha_z p_z c + \beta mc^2)\Psi(z) = (E - V(z))\Psi(z) \tag{1}$$

with

$$V(z) = -a\delta(z). \tag{2}$$

Note that translation invariance allows us to take the support of the δ at the origin without loss of generality. This is the potential normally used in the literature [5] by discussing the corresponding one-dimensional Schrödinger case.

We decompose the Dirac 4-spinor into upper and lower 2-spinor components $\psi_u(z)$ and $\psi_l(z)$, respectively:

$$\Psi(z) = \begin{bmatrix} \psi_u(z) \\ \psi_l(z) \end{bmatrix} \tag{3}$$

and integrate (1) in the range $-\varepsilon \leqslant z \leqslant +\varepsilon$ for a small $\varepsilon > 0$.

$$-i\hbar c\sigma_z \int_{-\varepsilon}^{+\varepsilon} \left(\frac{d\psi_l}{dz}\right) dz + mc^2 \int_{-\varepsilon}^{+\varepsilon} \psi_u(z)\, dz = E \int_{-\varepsilon}^{+\varepsilon} \psi_u(z)\, dz + a\psi_u(0) \tag{4a}$$

$$-i\hbar c\sigma_z \int_{-\varepsilon}^{+\varepsilon} \left(\frac{d\psi_u}{dz}\right) dz - mc^2 \int_{-\varepsilon}^{+\varepsilon} \psi_l(z)\, dz = E \int_{-\varepsilon}^{+\varepsilon} \psi_l(z)\, dz + a\psi_l(0). \tag{4b}$$

We have used the usual convention of Bjorken and Drell [6] for the Dirac matrices α_i in terms of the Pauli matrices and for the decomposition of the Dirac spinor.

If we take the limit $\varepsilon \to 0$ in the previous equations we get

$$-\mathrm{i}\hbar c\sigma_z(\psi_\mathrm{l}^+(0)-\psi_\mathrm{l}^-(0)) = a\psi_\mathrm{u}(0) \tag{5a}$$

$$-\mathrm{i}\hbar c\sigma_z(\psi_\mathrm{u}^+(0)-\psi_\mathrm{u}^-(0)) = a\psi_\mathrm{l}(0). \tag{5b}$$

In these expressions $\psi^+(0)$ and $\psi^-(0)$, denote the result of taking the limit $\varepsilon \to 0$ from the right and the left, respectively. Note that we have generated a discontinuity in the upper and lower components at $z=0$ in such a way that the discontinuity of one component is given by the value of the other component at this point. The problem now lies in finding the value of each component at the support of the δ. In order to answer this question we consider the bound-state solution, i.e. $|E|<m$, of (1) in the regions given by $z<0$ and $z>0$, where the potential vanishes. As in the non-relativistic discussion of the δ potential [5], we will have an exponential decaying wavefunction for a bound state, although the potential vanishes everywhere with the exception of the δ support. The corresponding Dirac spinors, as is well known, are given by the following.

Region I, z<0

$$\Psi(z) = A\begin{bmatrix}1\\0\\\mathrm{i}k/(E+m)\\0\end{bmatrix}\exp(kz) + B\begin{bmatrix}1\\0\\-\mathrm{i}k/(E+m)\\0\end{bmatrix}\exp(-kz). \tag{6}$$

Region II, z>0

$$\Psi^+(z) = C\begin{bmatrix}1\\0\\\mathrm{i}k/(E+m)\\0\end{bmatrix}\exp(kz) + D\begin{bmatrix}1\\0\\-\mathrm{i}k/(E+m)\\0\end{bmatrix}\exp(-kz). \tag{7}$$

We use for the moment the unit system $\hbar=c=1$. In this expression k is given by

$$k=(m^2-E^2)^{1/2}. \tag{8}$$

We must have $B=C=0$ in the previous equations in order to have a normalised state. Note that it is sufficient to look for one particular electron solution of the Dirac equation, with a definite spin polarisation, since the potential is spin independent.

The value of the wavefunction at $z=0$ can be found once we have a relation between the coefficients A and D. As will be shown, parity provides us with the required relation. Parity is a good quantum number for our problem since the parity operator commutes with the Hamiltonian. As is well known [6], the parity operator Π acts on the eigenstates of the Hamiltonian in the following way:

$$\Pi\psi(X,t) = \gamma^0\psi(-X,t) = \pm\psi(X,t). \tag{9a}$$

If we take the upper and lower components explicitly, the previous relation implies

$$\begin{bmatrix}\psi_\mathrm{u}(-z)\\-\psi_\mathrm{l}(-z)\end{bmatrix} = \pm\begin{bmatrix}\psi_\mathrm{u}(z)\\\psi_\mathrm{l}(z)\end{bmatrix}. \tag{9b}$$

We conclude that one of the components must be even and the other odd. This means that

$$A = \pm D. \tag{9c}$$

The plus sign denotes a Dirac spinor where the upper component is even and the lower component odd. The minus sign corresponds to the opposite situation. The insertion of the explicit expressions for $\psi_u^\pm$ and $\psi_l^\pm(0)$ in (5*a*) and (5*b*) gives us the energy of the bound state for each case. In the 'normal' unit system ($\hbar \neq 1$, $c \neq 1$) we have

$$E = mc^2 \frac{(4\hbar^2c^2 - a^2)}{(4\hbar^2c^2 + a^2)} \qquad \text{if } A = D \tag{10a}$$

$$E = -mc^2 \frac{(4\hbar^2c^2 - a^2)}{(4\hbar^2c^2 + a^2)} \qquad \text{if } A = -D. \tag{10b}$$

The existence of two energy eigenvalues in the interval $[m, -m]$ is consistent with general properties of the Dirac operator [7]. In figure 1 we show the function $E(a)$. Although from the mathematical point of view both solutions are valid, from these curves we may conclude that the first choice for the sign is the correct one because the energy of the bound state becomes more negative for an increasing value of the coefficient a, as it should be from an intuitive point of view. From (10*a*) we can see that our bound state will never dive into the Dirac sea. If $a \to \infty$ the ground state goes up to $E = -mc^2$, only touching the border of the sea. Therefore, we will not have supercritical effects in this case.

Note that the upper component of the Dirac spinor is continuous at $z = 0$ if $A = D$. All the discontinuity induced by the δ potential concentrates in the lower component and its value at $z = 0$ is given by the average of $\psi_l^\pm(0)$.

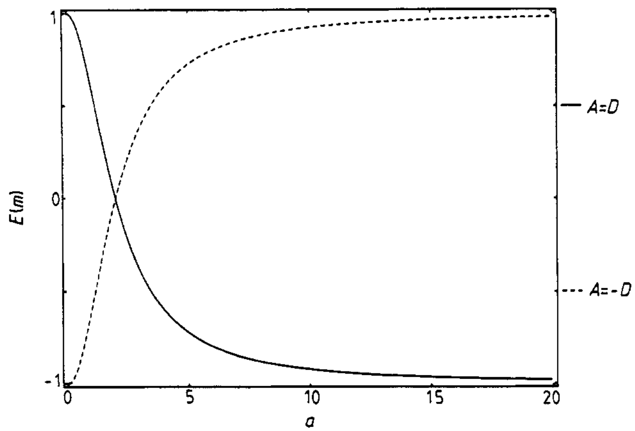

Figure 1. This figure shows the energy of the bound state, in units of m, for the one-dimensional problem, as a function of the δ coefficient a. The full curve corresponds to $A = D$ and the broken curve to $A = -D$.

3. The three-dimensional Dirac potential

In this section we want to discuss the case of a central spherically symmetric attractive δ potential. This potential, of course, depends only on the radial coordinate

$$V(r) = -a\delta(r - r_0). \tag{11}$$

It is convenient at this point to recall some general properties of the solution of the Dirac equation in a central potential. For more details the reader may consult the book by Greiner *et al* quoted in [1]. In this case, the complete set of commuting operators is given by H, J^2, J_3 and K, where the last one is defined by

$$K = \beta(\mathbf{\Sigma} \cdot \boldsymbol{L} + \hbar). \tag{12}$$

In this expression $\mathbf{\Sigma}$ denotes the usual spin operator in the Dirac theory and $\boldsymbol{L}$ is the angular momentum operator. The eigenvalues of the operator K are given by

$$K\Psi = -\kappa h\Psi = \pm(j+\tfrac{1}{2})\hbar\Psi. \tag{13}$$

In (18) $j = 0, \frac{1}{2}, 1, \frac{3}{2}$, etc, are the eigenvalues of J^2 according to the usual relation:

$$J^2\Psi = j(j+1)\hbar^2\Psi. \tag{14}$$

We note that the four-component Dirac spinor is not an eigenvalue of L^2. However the upper and lower components, taken separately, satisfy

$$L^2\psi_u(x) = [j(j+1)\hbar^2 + \kappa\hbar^2 + \tfrac{1}{4}\hbar^2]\psi_u(x) \equiv 1_u(1_u+1)\hbar^2\psi_u(x) \tag{15a}$$

$$L^2\psi_l(x) = [j(j+1)\hbar^2 - \kappa\hbar^2 + \tfrac{1}{4}\hbar^2]\psi_l(x) \equiv 1_l(1_l+1)\hbar^2\Psi_l(x). \tag{15b}$$

Note that the orbital parities of the upper and lower components have opposite sign. It is convenient to parametrise the four-component spinor by separating the radial and angular dependence, as follows:

$$\Psi = \begin{bmatrix} \psi_u \\ \psi_l \end{bmatrix} = \begin{bmatrix} g(r)\,Y_{l_u}^{j,j_3}(\theta, \Phi) \\ if(r)\,Y_{l_l}^{j,j_3}(\theta, \Phi) \end{bmatrix}. \tag{16}$$

For our present discussion we do not need the explicit form of the angular components. If we now consider the Dirac equation for a spinor parametrised in this way, it is not difficult to show that the radial components satisfy the following set of coupled differential equations:

$$\hbar c\left(\frac{\mathrm{d}F}{\mathrm{d}r} - \frac{\kappa F}{r}\right) = -(E - V(r) - mc^2)G(r) \tag{17a}$$

$$\hbar c\left(\frac{\mathrm{d}G}{\mathrm{d}r} + \frac{\kappa G}{r}\right) = (E - V(r) + mc^2)F(r) \tag{17b}$$

$F(r)$ and $G(r)$ are defined as follows:

$$\begin{aligned} G(r) &\equiv rg(r) \\ F(r) &\equiv rf(r). \end{aligned} \tag{18}$$

Let us consider (17a) and (17b) for our attractive δ-shell potential. If we integrate the differential equations between $r_0 - \varepsilon$ and $r_0 + \varepsilon$ taking then the limit $\varepsilon \to 0$ we find

$$\hbar c[F^+(r_0) - F(r_0)] = -aG(r_0) \tag{19a}$$

$$\hbar c[G^+(r_0) - G^-(r_0)] = aF(r_0). \tag{19b}$$

This result is independent of the measure used in the integral, at least for singularity-free weight functions. The previous equations show the appearance of discontinuities in the radial components of the wavefunctions. In this case, however, it is clear that parity will not help us to solve the problem, i.e. to fix the value of the upper and lower components at the support of the potential, since this operator does not connect the inside solution ($r < r_0$) with the outside solution ($r > r_0$).

In recent work by Dittrich *et al* [7] we can find a thorough discussion of the mathematical properties of Dirac operators with a spherically symmetric δ-shell interaction, i.e. precisely our case. The boundary conditions which must be satisfied by the inside and outside solutions at the δ support are established in theorem 3.2 of [7]. Let us take again the discontinuity equations (19*a*) and (19*b*) and assume that the values of the $G(r)$ and $F(r)$ at r_0 are given by a convex combination with an arbitrary parameter α, $0 \leqslant \alpha \leqslant 1$, of the form

$$G(r_0) = \alpha G^+(r_0) + (1-\alpha) G^-(r_0) \tag{20a}$$

$$F(r_0) = \alpha F^+(r_0) + (1-\alpha) F^-(r_0). \tag{20b}$$

If we take for the moment $(a/hc) = 1$, (19*a*) and (19*b*) can be written as

$$\begin{bmatrix} F^-(r_0) \\ G^-(r_0) \end{bmatrix} = \begin{bmatrix} \dfrac{1-\alpha(1-\alpha)}{1+(1-\alpha)^2} & \dfrac{1}{1+(1-\alpha)^2} \\ \dfrac{-1}{1+(1-\alpha)^2} & \dfrac{1-\alpha(1-\alpha)}{1+(1-\alpha)^2} \end{bmatrix} \begin{bmatrix} F^+(r_0) \\ G^+(r_0) \end{bmatrix}. \tag{21}$$

The quoted theorem establishes that, in order to have a self-adjoint Dirac Hamiltonian, the matrix A which relates the inside and outside solutions must be real and $\det A = 1$. If we impose the last condition we get as the only real solution $\alpha = \frac{1}{2}$, i.e. the same answer we found in the one-dimensional case through the parity analysis. Of course, we could have used in this case the same theorem, since (5) and (19) have the same shape.

The regular solutions in the two regions where the potential vanishes, $r < r_0$ and $r > r_0$, corresponding to bound states, i.e. states with $|E| < m$, are given by the following radial components.

Region I, $r < r_0$

$$G^-(r) = a_1 r \left(\frac{\pi}{2kr}\right)^{1/2} I_{l_\kappa + \frac{1}{2}}(kr) \tag{22a}$$

$$F^-(r) = \frac{a_1 k\hbar c}{E + mc^2} r \left(\frac{\pi}{2kr}\right)^{1/2} I_{l_{-\kappa} + \frac{1}{2}}(kr). \tag{22b}$$

Region II, $r > r_0$

$$G^+(r) = a_2 r \left(\frac{\pi}{2kr}\right)^{1/2} K_{l_\kappa + \frac{1}{2}}(kr) \tag{23a}$$

$$F^+(r) = -\frac{a_2 k\hbar c}{E + mc^2} r \left(\frac{\pi}{2kr}\right)^{1/2} K_{l_{-\kappa} + \frac{1}{2}}(kr). \tag{23b}$$

In these expressions $[\pi/2x]^{-1/2} I_{n+1/2}(x)$ and $[\pi/2x]^{-1/2} K_{n+1/2}(x)$ are the modified spherical Bessel functions [8]. In terms of κ, see (13), we have defined

$$\begin{aligned} l_\kappa &= \kappa && \text{if } \kappa > 0 \\ l_\kappa &= -\kappa - 1 && \text{if } \kappa < 0 \end{aligned} \tag{24a}$$

$$l_{-\kappa} = -\kappa \qquad \text{if } \kappa < 0$$
$$l_{-\kappa} = \kappa - 1 \qquad \text{if } \kappa > 0. \tag{24b}$$

In the case of the ground state, corresponding to an upper component spinor with $l_\kappa = 0$, (22) and (23) reduce to

$$I_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sinh(z)$$
$$K_{1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} \exp(-z) \tag{25}$$

and

$$I_{3/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \left(\cosh(z) - \frac{\sinh(z)}{z}\right)$$
$$K_{3/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} \exp(-z)(1 + 1/z). \tag{26}$$

If we substitute the averages of the radial functions at r_0 in the radial discontinuity equations, and eliminate the coefficients a_1 and a_2, we get a general expression for the energy eigenvalue equation. In the case of the ground state it assumes the form ($\hbar = c = 1$)

$$\tanh(kr_0) = \frac{akr_0(kr_0+1) - kr_0^2(E+m)(\frac{1}{4}a^2 - 1)}{a(E+m)^2 r_0^2 + a(kr_0+1) + kr_0^2(E+m)(\frac{1}{4}a^2-1)}. \tag{27}$$

If we solve this equation numerically for $r_0 = 1\ m^{-1}$, we find that the ground state disappears into the sea for $a_c \approx 3.7$ inducing positron emission for all values of a larger than a_c. From the mathematical point of view (see theorem 6.2 of [7]) at this point the corresponding wavefunctions are no longer square integrable. Beyond $[-m, m]$ we probably have resonances.

In figure 2 we show the dependence of the ground-state energy on the parameter a for $r_0 = 1\ m^{-1}$, $0.5\ m^{-1}$. Note that a smaller value of r_0 implies a more abrupt

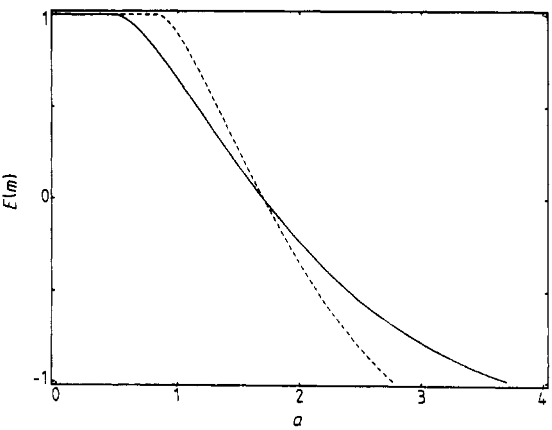

Figure 2. In this figure we represent the behaviour of the ground-state energy, in units of m, for the three-dimensional case, as a function of the δ coefficient a. The full curve corresponds to $r_0 = 1\ m^{-1}$ ($\hbar = c = 1$) and the broken curve to $r_0 = 0.5\ m^{-1}$.

behaviour of the ground-state energy as a function of a. If r_0 diminishes, a larger value of a is required to induce a bound state. However, once this bound state appears, it goes down more rapidly, as a function of a, to $E=-mc^2$ and dives into the sea. For example $a_c \approx 2.77$ for $r_0 = 0.5$. If r_0 becomes very small, what we really have is an unstable behaviour. The coefficient a needs to be large, but then the ground state dives into the sea extremely rapidly. In any case it is well known that a delta potential concentrated at one point in the three-dimensional case is completely trivial and does not affect the wavefunction at all [9].

To conclude this section we would like to make a remark concerning the non-relativistic behaviour of our solution. The structure of the spinor radial components $F(r)$ and $G(r)$, (25) and (26), tells us that the lower $F(r)$ component is supressed by an order $1/c$ with respect to the upper component $G(r)$. In the non-relativistic limit, where $c \to \infty$, the upper component becomes continuous and we have the usual Schrödinger solution to the δ-potential problem [5].

4. Conclusions

In this paper we have considered a curious situation in relativistic quantum mechanics, namely the discontinuities induced by a δ potential on the wavefunction and its possible physical consequences.

Supercritical effects will be absent in the one-dimensional situation with the δ potential concentrated at the origin. In the three-dimensional case, in contrast, the ground state dives into the Dirac sea for some critical value of the coefficient in front of the δ potential, inducing positron emission. From our analysis it is clear that these supercritical effects depend on the spatial extension of the potential. In fact, if the radius of the δ-shell potential becomes smaller it is more difficult to induce the existence of bound states, but they descend more quickly, as a function of the δ coefficient, into the depths of the sea.

Acknowledgments

The authors want to express their gratitude to R Tarrach, N Bralic and R Espinosa for their interest in this work and numerous critical comments. We acknowledge E Wyndham for a careful reading of the manuscript. Finally we are indebted to the referee of this paper for pointing us the existence of [7] and to R Benguria for helpful discussions.

This work was partially supported by DUC under grants 046-87, 048-87 and FONDECYT under grants 677-87 and 367-88.

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